Not So Fresh ML

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Towards safer and more expressive languages for meta-programming

Program representation should stay well-typed and well-scoped

Pursuing the work on FreshML

- Inspired from FreshML
- pure FreshML for its safety
- Cαml for its expressiveness

A taste of FreshML/Cαml

Data type for explicitly typed lambda calculus

```
data Term
  = Var Atom
  | App Term Term
  | Lam < Atom × neutral Ty × inner Term >
  | Let < Atom × outer Term × inner Term >
```

Capture avoiding substitution

```
subst :: (Atom, Term) → Term → Term
subst (a, v) = go
  where
    go (Var b)         = if a ≡ b then v else Var b
    go (App t u)       = App (go t) (go u)
    go (Lam<b,ty,t>) = Lam<b, ty, go t>
    go (Let<b,t,u>)   = Let<b, go t, go u>
```

Computing the size of a term

```
size :: Term → Int
size (Var _)          = 1
size (App t u)        = 1 + size t + size u
size (Lam<_,_,t>)     = 3 + size t
size (Let<_,t,u>)     = 3 + size t + size u
```

FreshML considered !

FreshML considered too fresh!

More efficient programs

Freshening is useless while:

- Computing the size of a term

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- Computing free variables

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- Typing some languages

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- Deciding α-equivalence

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- ...

To freshen or not to freshen?

- While FreshML implicitly freshen

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- This system allows both non-freshening and freshening openings

To freshen or not to freshen?

- While FreshML implicitly freshen
- This system allows both non-freshening and freshening openings
- However we will only the non-freshening part

World-index types for atoms

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let x = x in x

Let's classify atoms by a world they live in

The type of atoms is now indexed by a world

type Atom α

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The type of atoms is now indexed by a world

type Atom α

Equality is homogeneous and prevents mixing worlds

$(\equiv)_{\mathsf{Atom}} :: \forall\ \alpha.\ \mathsf{Atom}\ \alpha \rightarrow \mathsf{Atom}\ \alpha \rightarrow \mathsf{Bool}$

Data type for explicitly typed lambda calculus

$$
\begin{array}{l}
\mathsf{data}\ \ \mathsf{Term} \\
\quad = \mathsf{Var}\ \ \mathsf{Atom} \\
\quad \mid \mathsf{App}\ \ \mathsf{Term}\ \ \mathsf{Term} \\
\quad \mid \mathsf{Lam}\ < \mathsf{Atom}\ \times\ \mathsf{neutral}\ \ \mathsf{Ty}\ \times\ \mathsf{inner}\ \ \mathsf{Term}\ > \\
\quad \mid \mathsf{Let}\ < \mathsf{Atom}\ \times\ \mathsf{outer}\ \ \mathsf{Term}\ \times\ \mathsf{inner}\ \ \mathsf{Term}\ >
\end{array}
$$

Data type for explicitly typed lambda calculus

```
data Term outer
  = Var (Atom outer)
  | App (Term outer) (Term outer)
  | ∃inner. Lam (Atom inner) Ty (Term inner)
  | ∃inner. Let (Atom inner) (Term outer) (Term inner)
```

Data type for explicitly typed lambda calculus

$$
\begin{array}{l}
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\quad \mid \exists\ \beta.\ \mathsf{Lam}\ (\mathsf{Atom}\ \ \beta)\ \mathsf{Ty}\ (\mathsf{Term}\ \ \beta) \\
\quad \mid \exists\ \beta.\ \mathsf{Let}\ (\mathsf{Atom}\ \ \beta)\ (\mathsf{Term}\ \ \alpha)\ (\mathsf{Term}\ \ \beta)
\end{array}
$$

Worlds are closely related to each other

The type of (oriented) links between worlds

type $\beta \triangleright \alpha$

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Links holds the set of atoms as a frontier

$\alpha \overset{(\notin S)}{\triangleright} \beta$

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Links are supposed to be invisible/inferred!

Data type for explicitly typed lambda calculus

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\quad \mid \ \exists\ \beta.\ \mathsf{Lam}\ \ (\beta\ \triangleright\ \alpha)\ \ (\mathsf{Atom}\ \ \beta)\ \ \mathsf{Ty}\ \ (\mathsf{Term}\ \ \beta) \\
\quad \mid \ \exists\ \beta.\ \mathsf{Let}\ \ (\beta\ \triangleright\ \alpha)\ \ (\mathsf{Atom}\ \ \beta)\ \ (\mathsf{Term}\ \ \alpha)\ \ (\mathsf{Term}\ \ \beta)
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$$

Link operations

Identity link

$\mathsf{id}_{\mathsf{Link}} :: \forall\ \alpha.\ \alpha \triangleright \alpha$

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Identity link

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Link composition

$(\circ)_{\mathsf{Link}} :: \forall\ \alpha\ \beta\ \gamma.\ (\beta \rhd \gamma) \to (\alpha \rhd \beta) \to (\alpha \rhd \gamma)$

Link operations

Identity link

$\mathsf{id}_{\mathsf{Link}} :: \forall\ \alpha.\ \alpha \triangleright \alpha$

Link composition

$(\circ)_{\mathsf{Link}} :: \forall\ \alpha\ \beta\ \gamma.\ (\beta \triangleright \gamma) \rightarrow (\alpha \triangleright \beta) \rightarrow (\alpha \triangleright \gamma)$

Atomic link

$\mathsf{atomic}_{\mathsf{Link}} :: \forall\ \alpha.\ \mathsf{Atom}\ \alpha \rightarrow (\alpha \triangleright \alpha)$

Casts to walk through links

Atomic casts

$\mathsf{cast}_{\mathsf{Atom}} :: \forall\ \alpha\ \beta.\ (\beta \triangleright \alpha) \rightarrow (\mathsf{Atom}\ \beta \rightarrow \mathsf{Atom}\ \alpha)$

Casts to walk through links

Atomic casts

$\mathsf{cast}_{\mathsf{Atom}} :: \forall\ \alpha\ \beta.\ (\beta \triangleright \alpha) \to (\mathsf{Atom}\ \beta \to \mathsf{Atom}\ \alpha)$

Generalized casts

$\mathsf{cast}_{\mathsf{f}} :: \forall\ \alpha\ \beta\ \mathsf{f}.\ (\beta \triangleright \alpha) \to (\mathsf{f}\ \beta \to \mathsf{f}\ \alpha)$

Casts to walk through links

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Generalized casts

$\mathsf{cast}_{\mathsf{f}} :: \forall\ \alpha\ \beta\ \mathsf{f}.\ (\beta \triangleright \alpha) \to (\mathsf{f}\ \beta \to \mathsf{f}\ \alpha)$

Cast implies proof obligations or dynamic checks!

Atom abstraction as existential quantification

Hiding the real world but keeping a link

```
data α<f> = ∃ β. Abs (β ▷ α) (Atom β) (f β)
```

Data type for explicitly typed lambda calculus

$$
\begin{array}{l}
\mathsf{data}\ \mathsf{Term}\ \alpha \\
\quad = \mathsf{Var}\ (\mathsf{Atom}\ \alpha) \\
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\end{array}
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Data type for explicitly typed lambda calculus

$$
\begin{aligned}
&\mathsf{data\ Term}\ \alpha\\
&\quad = \mathsf{Var}\ (\mathsf{Atom}\ \alpha)\\
&\quad \mid \mathsf{App}\ (\mathsf{Term}\ \alpha)\ (\mathsf{Term}\ \alpha)\\
&\quad \mid \mathsf{Lam}\ \alpha{<}\lambda\beta{\rightarrow}\ (\mathsf{Ty},\ \mathsf{Term}\ \beta){>}\\
&\quad \mid \mathsf{Let}\ \alpha{<}\lambda\beta{\rightarrow}\ (\mathsf{Term}\ \alpha,\ \mathsf{Term}\ \beta){>}
\end{aligned}
$$

Making an abstraction

$\mathsf{Abs} :: \forall\ \alpha\ \beta\ \mathsf{f}.\ (\beta \rhd \alpha) \rightarrow \mathsf{Atom}\ \beta \rightarrow \mathsf{f}\ \beta \rightarrow \alpha\mathsf{<f>}$

Making an abstraction

$\mathsf{Abs} :: \forall\ \alpha\ \beta\ \mathsf{f}.\ (\beta \rhd \alpha) \rightarrow \mathsf{Atom}\ \beta \rightarrow \mathsf{f}\ \beta \rightarrow \alpha\text{<f>}$

$\mathsf{Lam} :: \forall\ \alpha\ \beta.\ (\beta \rhd \alpha) \rightarrow \mathsf{Atom}\ \beta \rightarrow \mathsf{Term}\ \beta \rightarrow \mathsf{Term}\ \alpha$

Making an abstraction

```
Abs :: ∀ α β f. (β ▷ α) → Atom β → f β → α<f>
```

```
Lam :: ∀ α β. (β ▷ α) → Atom β → Term β → Term α
```

```
mkLam :: ∀ α. Atom α → Term α → Term α
mkLam x t = Lam (atomic x) x t
```

Making an abstraction

$$\mathsf{Abs} :: \forall\ \alpha\ \beta\ \mathsf{f}.\ (\beta \rhd \alpha) \to \mathsf{Atom}\ \beta \to \mathsf{f}\ \beta \to \alpha\mathsf{<f>}$$

$$\mathsf{Lam} :: \forall\ \alpha\ \beta.\ (\beta \rhd \alpha) \to \mathsf{Atom}\ \beta \to \mathsf{Term}\ \beta \to \mathsf{Term}\ \alpha$$

$$\begin{array}{l} \mathsf{mkLam} :: \forall\ \alpha.\ \mathsf{Atom}\ \alpha \to \mathsf{Term}\ \alpha \to \mathsf{Term}\ \alpha \\ \mathsf{mkLam}\ \mathsf{x}\ \mathsf{t} = \mathsf{Lam}\ (\mathsf{atomic}\ \mathsf{x})\ \mathsf{x}\ \mathsf{t} \end{array}$$

$$\mathsf{mkConst}\ \mathsf{x}\ \mathsf{y} = \mathsf{mkLam}\ \mathsf{x}\ (\mathsf{mkLam}\ \mathsf{y}\ (\mathsf{Var}\ \mathsf{x}))$$

Opening an abstraction does not freshen it

```
let (Abs lnk x y) = t in u
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$\Gamma \vdash$ t : α<f> where $\alpha \in \Gamma$

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let (Abs lnk x y) = t in u
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$\Gamma \vdash$ t : α<f> where $\alpha \in \Gamma$

Γ,β,lnk:$\beta \triangleright \alpha$,x:Atom β,y:f $\beta \vdash$ u : τ where $\beta \# \tau$

Safety Properties

- Well-typed programs do not get stuck

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- Casts may dynamically fail or be proven successful
- α-equivalence is defined structurally on types

Example commuting abstraction with pairs

```
commute :: ∀ α. α<λβ→ (Term β, Term β)>
                 → (α<Term>, α<Term>)
commute t =
  let (Abs lnk x (y,z)) = t
  in (Abs lnk x y, Abs lnk x z)
```

Name capture does not type-check

```
wrong :: ∀ α. α<Term> → Term α → α<Term>
wrong t u =
  let (Abs lnk x y) = t
  in Abs lnk x u
```

Computing the size of a term

```
size :: ∀ α. Term α → Int
size (Var _)          = 1
size (App t u)        = 1 + size t + size u
size (Lam _ _ _ t)    = 3 + size t
size (Let _ _ t u)    = 3 + size t + size u
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```

Polymorphic recursion!

Computing free variables

```
remove :: Atom → [Atom] → [Atom]
remove _ [] = []
remove a (b:bs)
  | a ≡ b = remove a bs
  | otherwise = b : remove a bs
```

```
fv :: Term → [Atom]
fv (Var a)         = [a]
fv (App t u)       = fv t ++ fv u
fv (Lam<a,_,t>)    = remove a (fv t)
fv (Let<a,t,u>)    = fv t ++ remove a (fv u)
```

Computing free variables

```
remove ::
    ∀ β α. (β ▷ α) → Atom β → [Atom β] → [Atom α]
remove _    _ [] = []
remove lnk a (b:bs)
   | a ≡ b = remove lnk a bs
   | otherwise = cast lnk b : remove lnk a bs
```

```
fv :: ∀ α. Term α →  [Atom α]
fv (Var a)            = [a]
fv (App t u)          = fv t ++ fv u
fv (Lam lnk a _ t) = remove lnk a (fv t)
fv (Let lnk a t u) = fv t ++ remove lnk a (fv u)
```

Looking up an environment

```
data Env β = Empty
           | ∃ α. Snoc (β ▷ α) (Env α) (Atom β) Ty
```

```
lookupEnv :: ∀ α. Atom α → Env α → Ty
lookupEnv a (Snoc lnk env b ty)
  | a ≡ b      = ty
  | otherwise  = lookupEnv (cast lnk a) env
lookupEnv _ Empty = error "unbound value"
```

Typing a term

```
typing :: ∀ α. Env α → Term α → Ty
typing env (Var v)
  = lookupEnv v env
typing env (Lam lnk a ty t)
  = ty `TyArrow` typing (Snoc lnk env a ty) t
typing env (Let lnk a t u)
  = typing (Snoc lnk env a (typing env t)) u
typing env (App t u)
  = case typing env t of
      from `TyArrow` to | from ≡ typing env u → to
      _ → error "ill typed"
```

Challenges and future work

- Deeper formalization and proofs

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- Deeper formalization and proofs
- α-equivalence for inside-out abstractions
- Better understanding of heterogeneous comparison
- Integrating complex binding structures
- Properties implied by world polymorphism

Conclusion

- Explicit scopes using world indices

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- Non-freshening opening

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Conclusion

- Explicit scopes using world indices
- Non-freshening opening
- Atom abstraction as existential quantification
- Expressivness close to a manual model with names

Questions?

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```
data Term
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```

Data type for explicitly typed lambda calculus

```
data Term outer
  = Var (Atom outer)
  | App (Term outer) (Term outer)
  | ∃inner. Lam (Atom inner) Ty (Term inner)
  | ∃inner. Let (Atom inner) (Term outer) (Term inner)
```

Data type for explicitly typed lambda calculus

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\begin{array}{l}
\text{data Term } \alpha \\
\quad = \text{Var (Atom } \alpha) \\
\quad \mid \text{App (Term } \alpha) \text{ (Term } \alpha) \\
\quad \mid \exists\ \beta.\ \text{Lam (Atom } \beta) \text{ Ty (Term } \beta) \\
\quad \mid \exists\ \beta.\ \text{Let (Atom } \beta) \text{ (Term } \alpha) \text{ (Term } \beta)
\end{array}
$$

Data type for explicitly typed lambda calculus

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\begin{array}{l}
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\quad \mid \ \exists\ \beta.\ \mathsf{Lam}\ \ (\beta\ \triangleright\ \alpha)\ \ (\mathsf{Atom}\ \ \beta)\ \ \mathsf{Ty}\ \ (\mathsf{Term}\ \ \beta) \\
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\end{array}
$$

Data type for explicitly typed lambda calculus

```
data Term α
  = Var (Atom α)
  | App (Term α) (Term α)
  | Lam α<λβ→ (Ty, Term β)>
  | Let α<λβ→ (Term α, Term β)>
```

Polymorphic values represent closed terms

A more direct presentation of atom sorts

Generalizing Cαml data structures

Picking fresh atoms

```
fresh x in t
```

Picking fresh atoms

```
fresh x in t
```

- The atom can be used in the world you like

Picking fresh atoms

`fresh x in t` where $\mathtt{x}\#(\mathtt{t}\Downarrow)$

- The atom can be used in the world you like
- Same proof obligation as in pure FreshML

Picking fresh atoms (second version)

```
fresh x,lnkExp,lnkImp in t
```

Picking fresh atoms (second version)

fresh x,lnkExp,lnkImp in t

$\Gamma,\beta,$lnkExp:$\beta \triangleright \alpha,$lnkImp:$\alpha \triangleright \beta,$x:Atom $\beta \vdash$ t : τ
 where $\alpha \in \Gamma$, $\beta \# \tau$

Picking fresh atoms (second version)

```
fresh x,lnkExp,lnkImp in t
```

$\Gamma,\beta,\mathsf{lnkExp}{:}\beta\rhd\alpha,\mathsf{lnkImp}{:}\alpha\rhd\beta,\mathsf{x}{:}\mathsf{Atom}\ \beta \vdash \mathsf{t} : \tau$
where $\alpha \in \Gamma,\ \beta \# \tau$

- The fresh atom is in an existential world

Picking fresh atoms (second version)

```
fresh x,lnkExp,lnkImp in t
```

Γ,β,lnkExp:$\beta \triangleright \alpha$,lnkImp:$\alpha \triangleright \beta$,x:Atom $\beta \vdash$ t : τ
where $\alpha \in \Gamma$, $\beta \# \tau$

- The fresh atom is in an existential world
- Links are provided to import and export things

Picking fresh atoms (second version)

```
fresh x,lnkExp,lnkImp in t
```

$\Gamma,\beta,\text{lnkExp}:\beta\triangleright\alpha,\text{lnkImp}:\alpha\triangleright\beta,\text{x}:\text{Atom}\ \beta \vdash \text{t} : \tau$
$\quad \text{where}\ \alpha \in \Gamma,\ \beta \# \tau$

- The fresh atom is in an existential world
- Links are provided to import and export things
- Proof obligations relied to casts

Freshening is still available

```
let (Abs _lnk (fresh x) y) = t in u
```

Freshening is still available

```
let (Abs _lnk (fresh x) y) = t in u
```

Freshening allows to use the same world

$\Gamma \vdash$ t : α<f> where $\alpha \in \Gamma$

Γ,_lnk:$\alpha \triangleright \alpha$,x:Atom α,y:f $\alpha \vdash$ u : τ

Precise control over scopes

Having explicit world subsume:

- Cαml inner/outer/neutral annotations

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- Cαml pattern types/expression types distinction

Precise control over scopes

Having explicit world subsume:

- Cαml inner/outer/neutral annotations
- Cαml pattern types/expression types distinction
- FreshML/Cαml atom sorts

Safe heterogeneous comparison!

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$$\text{atmEqH} :: \forall\ \alpha\ \beta.\ \text{Atom}\ \alpha \rightarrow \text{Atom}\ \beta \rightarrow (\beta \triangleright \alpha) \rightarrow \text{Bool}$$

Safe heterogeneous comparison!

$$\mathsf{atmEqH} :: \forall\ \alpha\ \beta.\ \mathsf{Atom}\ \alpha \rightarrow \mathsf{Atom}\ \beta \rightarrow (\beta \triangleright \alpha) \rightarrow \mathsf{Bool}$$

$$\begin{aligned} \mathsf{atmEqH}\ \mathsf{a}\ \mathsf{b}\ \mathsf{lnk}\ &|\ \mathsf{b} \notin \mathsf{lnk} &&= \mathsf{a} \equiv \mathsf{cast}\ \mathsf{lnk}\ \mathsf{b} \\ &|\ \mathsf{otherwise} &&= \mathsf{False} \end{aligned}$$

Substituting closed terms for variables

```
substClosed ::
    ∀ α. Atm α → (∀ β. Term β) → Term α → Term α
substClosed a v = go id
  where
    go :: ∀ δ. (δ ▷ α) → Term δ → Term δ
    go lnk (Var b) | atmEqH a b lnk = v
                   | otherwise      = Var b
    go lnk (App t u)
      = App (go lnk t) (go lnk u)
    go lnk (Lam lnk' b ty t)
      = Lam lnk' b ty (go (lnk ∘ lnk') t)
    go lnk (Let lnk' b t u)
      = Let lnk' b (go lnk t) (go (lnk ∘ lnk') u)
```